ON THE CONSTRUCTION OF PERIODIC MOTIONS

(O POSTROENII PERIODICHESKIKH DVIZHENII)

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Conditions are given under which one can select the input signal of a system so that a given periodic process may be generated approximately. The problem investigated here is related to the theory of programmed control [1, p. 231] since it considers the possibility of finding programming functions which will guarantee a stable periodic programming process. Under realistic conditions the programming functions can be given only approximately. This article presents estimates of the admissible errors of the required programming functions. From the strictly mathematical viewpoint, the problem can be reduced to the formulation of the conditions for stability of the periodic motion in the presence of constantly acting disturbances bounded in norm. An estimate of the absolute value, of the mean absolute value, and of the mean-square value of the above-indicated admissible error is given. The case when the constructed periodic motion is discontinuous is also considered. A part of the basic results is carried over to the case of a non-periodic approximate motion.

1. Let us consider the differential equation

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n, t) + \varphi_i(t) \qquad (i = 1, \ldots, n) \tag{1.1}$$

Under the assumption that all the functions $f_i(x_1, \ldots, x_n, t)$ are periodic functions of time with period ω, we set ourselves the problem of selecting such periodic functions $\phi_i(t)$ that a given system of periodic functions $x_i = \psi_i(t)$ $(i = 1, \ldots, n)$ of period ω may be a solution of the system (1.1). The solution of this simple problem has the form

$$\varphi_i(t) = \psi_i'(t) - f_i(\psi_1(t), \ldots, \psi_n(t), t) \qquad (i = 1, \ldots, n) \tag{1.2}$$

In practice it may, however, turn out to be entirely inapplicable. Indeed, if the system of the functions $x_i = \psi_i(t)$ defines some programming process, then this process can be realized only in case the process is stable relative to the initial disturbances. Furthermore, under real conditions, the programming functions $\phi_i(t)$ are selected from some narrowly prescribed class of functions, for example, from the class of polynomials, trigonometric polynomials, piece-wise constant discontinuous functions, and so on.

Equations (1.2) can, therefore, be satisfied only approximately with a certain error. In case the programming functions $\phi_i(t)$ are expressed as linear combinations of functions from a system of mutually orthogonal functions, it is simple to compute the mean-square error of the approximation. It is also known that a knowledge of the absolute value of the error makes it possible to obtain an estimate of the mean value and of the mean-square value of the error, while the knowledge of the mean-square value of the error permits one to obtain an estimate of its mean value.

Thus, the investigation of the problem on the preservation of the stability of the periodic motion of the system (1.1) under constant disturbances, bounded in the mean or in their mean-square value, is of special interest. The study of disturbances, which are bounded in the mean, can be carried over to the case of shocks or δ-type of disturbances, as will be shown in the sequel.

For the general case, the stability of motion under constantly acting disturbances bounded in the mean was considered in the work of Germaidze and Krasovskii [2]. Questions on the stability of the periodic motion under constantly acting disturbances bounded in their absolute value were considered in [3,4].

Problems on the existence and on the preservation and stability of periodic motion bounded in the modulus of external forces were considered in [5,6] on the basis of Liapunov's function. In the present article, estimates of the absolute mean and mean-square values of the admissible error of the approximating programming functions are obtained in a different way. Here, the approximating periodic motion Γ has the following properties.

1) All trajectories which start for $t = t_0$ in a small enough neighborhood of Γ do not leave an ϵ-neighborhood of Γ when $t > t_0$.

2) In the ϵ-neighborhood of Γ there exists an asymptotically stable periodic motion whose region of attraction contains some neighborhood of Γ.

Therefore, if the error of the approximation lies within admissible bounds, then the presence of a small enough error in the choice of the initial conditions will not prevent the approximate realization of the periodic process, since the obtained process will approach asymptotically a periodic motion close to the assigned one.

2. Let us assume first that the functions $\psi_i(t)$, which determine the periodic motion Γ, are continuous and piece-wise differentiable.

Making the following change of variables in the system (1.1)

$$z_i = x_i - \psi_i(t) \qquad (i = 1, \ldots, n)$$

we obtain

$$\frac{dz_i}{dt} = f(z_1 + \psi_1(t), \ldots, z_n + \psi_n(t), t) + \varphi_i(t) - \varphi_i'(t) \qquad (i = 1, \ldots, n)$$

Introducing the notation

$$Z_i(z_1, \ldots, z_n, t) = f_i(z_1 + \psi_1(t), \ldots, z_n + \psi_n(t), t) - f_i(\psi_1(t), \ldots, \psi_n(t), t)$$
$$r_i(t) = \varphi_i(t) - \psi_i'(t) + f_i(\psi_1(t), \ldots, \psi_n(t), t) \qquad (i = 1, \ldots, n)$$

we can express the system (1.1) in the form

$$\frac{dz_i}{dt} = Z_i(z_1, \ldots, z_n, t) + r_i(t) \qquad (i = 1, \ldots, n) \tag{2.1}$$

Equations (2.1) represent, obviously, the system of equations of the disturbed motion; the functions $r_i(t)$ determine the error of approximation of the approximating functions ϕ_i, while the deviation from zero of the solution $z_i(t)$ of the system (2.1) coincides with the deviation of the solution $x_i(t)$ of the system (1.1) from the assigned periodic motion.

By first separating in some way the linear part from the function $Z_i(z_1, \ldots, z_n, t)$, we can write (2.1) in the form

$$\frac{dz_i}{dt} = \sum_{k=1}^{n} a_{ik}(t) z_k + R_i(z_1, \ldots, z_n, t) + r_i(t) \qquad (i = 1, \ldots, n) \tag{2.2}$$

or in the matrix-vector form as

$$dz/dt = A(t)z + R(z, t) + r(t) \tag{2.3}$$

Let us determine the norm of the vector z and the norm of the matrix A by the following relation*:

* The norm of the vector z may be defined by any other of the known methods, and the norm of the matrix A can then be defined by the relation $\|A\| = \max \|Az\|$ when $\|z\| = 1$. In this case, all preceding remarks remain valid [7, p. 111].

$$\|z\| = \max_{1 \leqslant i \leqslant n} |z_i|, \qquad \|A\| = \max_{1 \leqslant i \leqslant n} (|a_{i1}| + \ldots + |a_{in}|)$$

Let D be the region given by the inequalities $\|z\| < \epsilon$, $0 \leqslant t < \infty$, and let us impose the following restrictions on the system (2.3):

a) The functions $A(t)$, $R(z, t)$ and $r(t)$ are periodic in t and of period ω.

b) In the region D, the function $R(z, t)$ satisfies the Lipschitz condition

$$\|R(z,t) - R(y,t)\| \leqslant L\|z-y\|, \qquad z \subset D, \qquad y \subset D$$

c) For a fixed z, the functions $a_{ik}(t)$ and $R_i(z, t)$ are Lebesgue integrable in absolute value on the interval $[0, \omega]$.

d) The functions $r_i^2(t)$ are Lebesgue integrable on the interval $[0, \omega]$.

e) There exists a fundamental matrix $W(t, \tau)$ of the system $z' = A(t)z$, which satisfies the conditions $W(\tau, \tau) = E$ (E is the unit matrix)

$$\|W(t,\tau)\| \leqslant Be^{-\alpha(t-\tau)}, \qquad B \geqslant 1, \qquad \alpha > 0$$

f) The quantity

$$\lambda = \alpha - LB > 0$$

We call attention to the fact that in view of a theorem of Caratheodory [8, p. 120] the conditions (b), (c) and (d) guarantee the existence of a unique solution of the system (2.3) in the region D.

Let us introduce the notation $\rho(t) = \|r(t)\|$.

Theorem 2.1. Let the conditions (a) to (f) and one of the following conditions be satisfied:

(A) $$\rho_0 = \sup_{0 \leqslant t \leqslant \omega} \rho(t) < \frac{\varepsilon}{2B^2}\lambda$$

(B) $$\rho_1 = \int_0^\omega \rho(t)\,dt < \frac{\varepsilon}{2B^2} e^{-\lambda\omega}(1 - e^{-\lambda\omega})$$

(C) $$\rho_2 = \left(\int_0^\omega \rho^2(t)\,dt\right)^{1/2} < \frac{\varepsilon}{2B^2}\left(\frac{2\lambda}{e^{2\lambda\omega} - 1}\right)^{1/2}(1 - e^{-\lambda\omega})$$

Let $\delta = \epsilon/2B$. Then the following assertions are true:

1) Every solution $z(t)$ of the system (2.3) which is such that $\| z(t_0) \| < \delta$ will not leave the region D if $t > t_0 \geqslant 0$.

2) In the region D, there exists an asymptotically stable periodic trajectory which attracts all other trajectories that come out of the region $\| z \| < \delta$, $t \geqslant 0$.

We shall now prove the theorem. Obviously, without destroying the generality, we may assume that $t_0 = 0$. By Cauchy's formula we have

$$z(t) = W(t, 0) z_0 + \int_0^t W(t, \tau)(R(z, \tau) + r(\tau))\, d\tau \tag{2.4}$$

Hence, on the basis of (b) and (e) we obtain

$$\| z(t) \| < Be^{-\alpha t} \| z_0 \| + B \int_0^t e^{-\alpha(t-\tau)} (L \| z \| + \rho(\tau))\, d\tau \tag{2.5}$$

Setting $u(t) = e^{\alpha(t)} \| z(t) \|$, we rewrite (2.5) in the form

$$u(t) < B \| z_0 \| + B \int_0^t (Lu(\tau) + e^{\alpha\tau} \rho(\tau))\, d\tau \tag{2.6}$$

From this it follows in accordance with Lemma 1.1 of [9] that

$$u(t) < Be^{BLt} \left(\| z_0 \| + \int_0^t \rho(\tau) e^{(\alpha - BL)\tau} d\tau \right) \tag{2.7}$$

Hence

$$\| z(t) \| < \Phi_1(t) + \Phi_2(t) \tag{2.8}$$

where

$$\Phi_1(t) = Be^{-\lambda t} \| z_0 \|, \qquad \Phi_2(t) = Be^{-\lambda t} \int_0^t \rho(\tau) e^{\lambda\tau} d\tau$$

Assuming that $\| z_0 \| < \delta = \epsilon/2B$, we have $\Phi_1(t) < \epsilon/2$ if $t > 0$. We shall show that if one of the conditions (A), (B) or (C) is satisfied then $\sup \Phi_2(t) < \delta$ if $t \geqslant 0$.

Suppose that condition (A) holds. In this case we have

$$\Phi_2(t) < \frac{\varepsilon}{2B} \lambda e^{-\lambda t} \int_0^t e^{\lambda\tau} d\tau$$

Since

$$e^{-\lambda t}\int_0^t e^{\lambda\tau}d\tau < \frac{1}{\lambda}$$

we deduce at once the required result.

Suppose that condition (B) holds, and let $t = k\omega + \tau_0$, $0 \leqslant \tau_0 < \omega$. In this case we have

$$\Phi_2(t) < Be^{-k\lambda\omega}[e^{\lambda\omega} + \ldots + e^{(k+1)\omega}]\rho_1 < B\frac{e^{\lambda\omega}}{1-e^{-\lambda\omega}}\rho_1.$$

This yields the required result.

Suppose, finally, that condition (C) holds. Then, obviously we have

$$\Phi_2(t) < Be^{-k\lambda\omega}\sum_{i=0}^{k}\left(\int_{i\omega}^{(i+1)\omega} e^{2\lambda\tau}\,d\tau\right)^{1/2}\rho_2 < B\left(\frac{e^{2\lambda\omega}-1}{2\lambda}\right)^{1/2}\frac{1}{1-e^{-\lambda\omega}}\rho_2$$

which again yields the required result.

And thus, if one of the conditions (A), (B) or (C) is satisfied, we find that $\sup \Phi_2(t) < \delta$ if $t > 0$, and, furthermore, that $\Phi_1(t) < \epsilon/2$. Hence, if $t > 0$ we have $\| z(t) \| < \delta + \epsilon/2 < \epsilon$, which proves the first part of the theorem.

In order to establish the existence of a periodic solution we must give a large enough number $N > 1$ such that $\Phi_2(t) \leqslant \delta(N-1)/N$. Since $\sup \Phi_2(t) < \delta$, it is obvious that such a number N always exists. Next we find a number

$$T = m\omega \geqslant \lambda^{-1}\ln BN$$

where m is a positive number.

Since $\Phi_1(T) \leqslant \delta/N$ if $t > 0$, it follows that $\| z(T) \| < \delta$ if $\| z(0) \| \leqslant \delta$.

Hence, the mapping $z = z(T)$ transforms the region $\| z \| < \delta$ into a part of itself. In order to apply now a well-known principle of contraction mappings [10, p. 90], we consider two points z_0 and y_0 in the region $\| z \| \leqslant \delta$. The difference between two solutions of the system (2.3) determined by these points satisfies the integral equation

$$z(t) - y(t) = W(t,0)(z_0 - y_0) + \int_0^t W(t,\tau)(R(z,\tau) - R(y,\tau))\,d\tau$$

From this we obtain the inequality

$$\|z(t)-y(t)\| \leqslant Be^{-\alpha t}\|z_0-y_0\| + BL\int_0^t e^{-\alpha(t-\tau)}\|z(\tau)-y(\tau)\|\,d\tau \qquad (2.9)$$

Introducing the notation $u(t) = \|z(t) - y(t)\| e^{\alpha t}$, we obtain

$$\|z(t)-y(t)\| < B\|z_0-y_0\| + BL\int_0^t u(\tau)\,d\tau$$

From this and from a known lemma [9] it follows that

$$u(t) < B\|z_0-y_0\| e^{BLt}$$

Thus, we finally obtain

$$\|z(t)-y(t)\| < B\|z_0-y_0\| e^{-\lambda t} \qquad (2.10)$$

and since $Be^{-\lambda t} \leqslant 1/N$

$$\|z(T)-y(T)\| < \frac{1}{N}\|z_0-y_0\|$$

This shows that the conditions for the applicability of the principle of contraction mappings are satisfied. Hence, there exists in the region $\|z\| < \delta$ a unique point y_0 such that $y(T) = y(0) = y_0$. This point determines for us the required periodic motion. Since $y(\omega) = y(\omega + T)$, it follows that the point $y(\omega)$, being a fixed point, must coincide with $y(0)$. Hence, the period of $y(t)$ is ω. The asymptotic stability of $y(t)$ follows from (2.10).

3. Let us next consider the case when the periodic motion $x_i = \psi_i(t)$ which is to be realized may have a finite number of discontinuities of the first kind. In this case the approximating programming functions must have the form

$$\varphi_i(t) = \psi_i'(t) - f_i(\psi_1(t), \ldots, \psi_n(t), t) \qquad (i = 1, \ldots, n) \qquad (3.1)$$

at those points where the derivative $\psi_i'(t)$ exists, and

$$\varphi_i(t) = \eta_{ik}\delta(t - t_k) \qquad (3.2)$$

at the discontinuity points $t_1, \ldots, t_m$. Here η_{ik} is the saltus of the function $\phi_i(t)$ at $t = t_k$; $\delta(t - t_k)$ is the Dirac function.

Under the assumption that the function $\phi_i(t)$ will be approximated by functions of the same type, let us express the error in the approximation in the form

$$r_i(t) = r_i^\circ(t) + \sum_{k=1}^{m} r_{ik}\delta(t - t_k) \qquad (i = 1, \ldots, n) \tag{3.3}$$

where $r_i^\circ(t)$ is a function whose absolute value is integrable on $[0, \omega]$.

Making use of the change of variables $z_i = x_i - \psi_i(t)$ in the system (1.1), we are again led to the system (2.1), and by separating in some way the linear part we obtain again the system (2.3). It should, however, be mentioned that even in the case when the functions $f_i(x_1, \ldots, x_n, t)$ of the system (1.1) are infinitely often differentiable with respect to $x_1, \ldots, x_n$, the matrix $A(t)$ and the function $R(z, t)$ may turn out to be discontinuous functions. In the more general case of the determination of the first approximation system and of the solution of the stability problem of this system, one should take into consideration the results obtained by Aizerman and Gantmakher [11].

Let

$$\rho(t) = \rho^\circ(t) + \sum_{k=1}^{m} \gamma_k\delta(t - t_k) \tag{3.4}$$

where

$$\rho(t) = \max_{1 \leqslant i \leqslant n} |r_i^\circ(t)| \qquad \text{for } t \neq t_k, \qquad \gamma_k = \max_{1 \leqslant i \leqslant n} |r_{ik}|$$

Theorem. Suppose that the conditions (a), (b), (c), (e) and (f) are satisfied and that the functions $r_i(t)$ are representable in the form (3.3). Let $\delta = \epsilon/2B$, and let us suppose that the inequality

$$\rho_1 = \int_0^{\omega - 0} \rho(t)\, dt = \int_0^{\omega} \rho^\circ(t)\, dt + \sum_{k=1}^{m} \gamma_k < \frac{\varepsilon}{2B^2} e^{-\lambda\omega}(1 - e^{-\lambda\omega})$$

is valid.

Under these hypotheses both assertions of Theorem 2.1 are true.

The proof of Theorem 3.1 is essentially a repetition of the proof of Theorem 2.1. Indeed, Formula (2.4) obviously applies in the present case if one makes use of the rule for the integration of expressions involving the Dirac function.

The only questionable step is the transition to the limit from the inequality (2.6) to the inequality (2.7). Let us show that this step is valid. Since for $t = 0$, the inequality (2.7) is true, and since both sides of the inequality are continuous from the right, this inequality will be valid if $0 \leqslant t < h$, where h is some positive number.

Let τ_0 be the infimum of the numbers t for which (2.7) is not true. Because of the fact that both sides of (2.7) are continuous from the right, we have

$$u(\tau_0) \geqslant Be^{BL\tau_0}\left(\|z_0\| + \int_0^{\tau_0} \rho(\tau)\, e^{(\alpha - BL)\tau} d\tau\right) \tag{3.5}$$

On the other hand, setting $t = \tau_0$ in (2.6), and replacing $u(t)$ under the integral sign by a larger quantity from (2.7), we obtain

$$u(\tau_0) < Be^{BL\tau_0}\|z_0\| + B^2L \int_0^{\tau_0} e^{BLt} \int_0^t \rho(\tau)\, e^{(\alpha - BL)\tau}\, dt + B\int_0^{\tau_0} e^{\alpha\tau}\rho(\tau)\, d\tau$$

Applying to the second integral the rule of differentiation by parts (which is valid under the given conditions, since we are dealing essentially with a Stiel'tjes integral

$$\int_0^{\tau_0} e^{\alpha\tau}\rho(\tau)\, d\tau$$

and since the integral exists), we obtain an inequality which contradicts the inequality (3.5).

From here on the proof of Theorem 3.1 is an exact repetition of the proof of Theorem 2.1.

We call attention to the fact that Theorem 3.1 can be formulated so that it will apply to the more general case when the programming functions $\phi_i(t)$ are of bounded variation. In this case, one should consider the equation

$$z(t) = W(t, t_0)\, z_0 + \int_{t_0}^t W(t, \tau)\, R(z, \tau)\, d\tau + \int_{t_0}^t W(t, \tau)\, dG$$

where the second integral is a Stieltjes integral with the integrating function $G(t)$ $(G_1(t), \ldots, G_n(t))$ being of bounded variation. This can be reduced to the previous case if one introduces the generalized functions $r_i(t)$ by defining them as $r_i(t) = G_i'(t)$ at the points where the derivative exists, and as $r_i(t) = r_{ik}\delta(t - t_k)$ at the points of discontinuity of $G_i(t)$ (r_{ik} is the saltus at the discontinuity). In the present case, the set of points of discontinuities of $G_i(t)$ may be infinite (but denumerable).

An even more general approach to the problem is possible on the basis of the concept of a generalized differential equation which is based on Kurtsveil's [12] generalization of the Perron integral.

4. Making use of the method of proof of Theorem 2.1 and of an idea expressed in [13], one can carry over some of the results of the indicated theorem to the case when the system (1.1), as well as the programming functions $x_i = \psi_i(t)$, are not periodic.

Let ω be an arbitrary positive number, and let

$$\rho(t) = \|r(t)\|, \qquad h_0 = \sup_{0 \leqslant t \leqslant \infty} \rho(t)$$

$$h_1 = \sup_{0 \leqslant t < \infty} \int_0^{t+\omega} \rho(t)\,dt, \qquad h_2 = \sup_{0 \leqslant t < \infty} \left(\int_t^{t+\omega} \rho^2(t)\,dt \right)^{1/2}$$

Theorem 4.1. Suppose that the conditions (b), (c), (d), (e) and (f) are satisfied (except that the integrability of the corresponding functions holds on any segment t, $t + \omega$, $t \geqslant 0$), and suppose that at least one of the following inequalities is valid:

$$(\mathrm{A}') \qquad h_0 < \frac{\varepsilon}{2B}\lambda$$

$$(\mathrm{B}') \qquad h_1 < \frac{\varepsilon}{2B} e^{-\lambda\omega}(1 - e^{-\lambda\omega})$$

$$(\mathrm{C}') \qquad h_2 < \frac{\varepsilon}{2B}\left(\frac{2\lambda}{e^{2\lambda\omega} - 1}\right)^{1/2}(1 - e^{-\lambda\omega})$$

Then every solution $z(t)$ of the system (2.3) which satisfies the condition

$$\|z(0)\| \leqslant \varepsilon/2B$$

will not leave the region D when $t \geqslant 0$.

The proof of Theorem 4.1 is the same as the proof of the first part of Theorem 2.1, except for the difference that in place of the inequality $\Phi_2(t) < \epsilon/2B$ one has to have the inequality

$$\Phi_2(t) < \frac{\varepsilon}{2}$$

Finally, let us consider the case when the approximating (nonperiodic) motion $x_i = \psi_i(t)$ has isolated discontinuities of the first kind. In this case, we again assume that $r_i(t)$ can be represented in the form (3.3), and we define the generalized function $\rho(t)$ in accordance with (3.4).

Let

$$h_1 = \sup_{0 \leqslant t < \infty} \int_t^{t+\omega-0} \rho(t)\, dt$$

Repeating the arguments used in the proof of Theorems 2.1 and 3.1, we establish that if

$$h_1 < \varepsilon/2Be^{-\lambda\omega}(1 - e^{-\lambda\omega})$$

the solution $z(t)$ of the system (2.3), under the condition that $\| z(0) \| < \epsilon/2B$, will not leave the region D when $t > 0$.

Thus, also in this last case, we can realize the desired programming process approximately with a accuracy of ϵ.

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